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GROUP CLASSIFICATION OF THE EQUATIONS OF TWO-DIMENSIONAL MOTIONS OF A GAS†

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A generalization of the definition of an equivalence group is proposed and a group classification of a system of equations describing the two-dimensional flows of an ideal gas [1] is given. Plane flows, which were earlier investigated from the group point of view in [2–4], are a special case of such motions. The algebraic approach employed rests on an analysis which has recently been developed [5, 6].

1. THE FACTOR SYSTEM

Solutions of the equations of gas dynamics invariant under the operator X_1 from the optimum system of sub-algebras of the algebra L_{11} [5] are considered. For simplicity in comparing our results with existing results for plane flows, our discussion will be carried out for a similar operator $X_3=\partial_z$, the invariants of which are $t, x, y, u, \upsilon, w, \rho, p$. The usual notation will be employed, namely, t is the time, (x, y, z) are spatial coordinates, $U=(u, \upsilon, w)$ is the velocity, ρ is the density, and p is the pressure of the gas. the invariant solution has the representation

$$U=U(t,x,y),\quad \rho=\rho(t,x,y),\quad p=p(t,x,y)$$

while the factor system is

$$\begin{aligned}&d_1U+\rho^{-1}\nabla_1 p=0,\quad d_1\rho+\rho\operatorname{div}_1 U=0\\&d_1p+A(p,\rho)\operatorname{div}_1 U=0\\&(d_1=\partial_t+u\partial_x+\upsilon\partial_y,\quad \nabla_1=(\partial_x,\partial_y,0),\quad \operatorname{div}_1 U=u_x+\upsilon_y)\end{aligned}\tag{1.1}$$

Particular solutions of system (1.1) with $w=0$ describe plane-parallel flows of a gas in the $R^2(x, y)$ plane.

2. THE EQUIVALENCE GROUP

For problems of group classification it is important to determine those transformations which change arbitrary elements, which are contained in the equations, while preserving the differential structure of the equations themselves.

Here we will give a construction of a group of equivalence transformations which is wider than that used in [7]. A generalization is achieved by including arbitrary elements in all

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coordinates of an infinitesimal operator, by means of which an equivalence group is set up. The possibility of choosing a representative in a group classification is then widened. For simplicity in describing the idea, the discussion will be carried out using the example of the differential equation

$$F(x, u, \phi, u_x) = 0 \tag{2.1}$$

where $x=(x, y)$ are the independent variables, $u=u(x)$ is the required function, and $\phi=\phi(u, x)$ is an arbitrary element. We will then seek a single-parameter group of continuous transformations

$$x'=f^x(x,u,\phi;a),\quad u'=f^u(x,u,\phi;a),\quad \phi'=f^\phi(x,u,\phi;a) \tag{2.2}$$

with group parameter a and infinitesimal operator

$$X^e=\xi^x\partial_x+\xi^y\partial_y+\zeta^u\partial_u+\zeta^\phi\partial_\phi \tag{2.3}$$

We will assume that all the coordinates of the infinitesimal operator X^e depend on (x, u, ϕ) unlike the approach used in [7], where this only applies to the component ζ^ϕ. The functions $\phi(u, x)$ and $u(x)$ act in different spaces, and hence, before writing the formulae for the coordinates of the continued operator X^e one must understand how the functions $\phi(u, x)$ and $u(x)$ are transformed by the action of the group (2.2), on which the following constraint is imposed. Any solution $u_0(x)$ of Eq. (2.1) with the function $\phi(u, x)$ when acted upon by (2.2) converts once more into the solution of an equation of the form (2.1), but with another (converted) function $\phi_a(u, x)$ which is defined in the usual way. Thus, by solving the following relations for (x, u)

$$x'=f^x(x,u,\phi(u,x);a),\quad u'=f^u(x,u,\phi(u,x);a)$$

we obtain

$$x=g^x(x',u';a),\quad u=g^u(x',u';a) \tag{2.4}$$

after which the transformed function $T_a(\phi)$ is determined

$$\phi_a(u',x')=f^\phi(x,u,\phi(x,u);a)$$

where, instead of (x, u) we have substituted their expressions (2.4). The transformed solution $T_a(u)=u_a(x)$ is obtained by solving the relations

$$x'=f^x(x,u_0(x),\ \phi(u_0(x),x);a)$$

for $x=\psi^x(a';a)$ and substituting these solutions into

$$u_a(x')=f^u(u,u_0(x),\ \phi_a(u_0(x),x);a)$$

Lemma. The transformations $T_a(u)$ constructed in this way form a group.

Proof. Since (2.2) forms a single-parameter group of continuous transformations, then, by the method of construction, the equality $T_b(T_a(\phi))=T_{a+b}(\phi)$ is satisfied. Taking this property into account, equating $T_b(T_a(u))$ and $T_{a+b}(u)$ we complete the proof of the lemma.

In agreement with the construction, the extended operator

$$\overline{X}^e = X^e + \zeta^{u_x}\partial_{u_x} + \zeta^{u_y}\partial_{u_y} + \zeta^{\phi_u}\partial_{\phi_u} + \zeta^{\phi_x}\partial_{\phi_x} + \zeta^{\phi_y}\partial_{\phi_y} + \ldots$$

has the following coordinates, which are connected with the dependent functions

$$\zeta^{u_\lambda} = D_\lambda^e\zeta^u - u_x D_\lambda^e \xi^x - u_y D_\lambda^e \xi^y$$

$$D_\lambda^e = \partial_\lambda + u_\lambda\partial_u + (\phi_u u_\lambda + \phi_\lambda)\partial_\phi$$

Here λ takes the values x and y. The coordinates of the extended operator $\overline{X}^e$, connected with an arbitrary element, are defined by the formulae

$$\zeta^{\phi_\lambda} = \tilde{D}_\lambda^e\zeta^\phi - \phi_x\tilde{D}_\lambda^e\xi^x - \phi_y\tilde{D}_\lambda^e\xi^y - \phi_u\tilde{D}_\lambda^e\zeta^u$$

$$\tilde{D}_\lambda^e = \partial_\lambda + \phi_\lambda\partial_\phi \quad (\lambda = u, x, y)$$

To construct the equivalence group of Eqs (2.6) we need to obtain the group (2.7) admissible by it with the extended operator $\overline{X}^e$ constructed above. Here we must take into account possible special, previously known properties of the arbitrary elements (for example, $\phi_x = 0$).

We can take as one of the examples of the extension of the equivalence group in this approach the group for the system of two equations with two independent variables [8]

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}p(u,\upsilon) = 0, \quad \frac{\partial \upsilon}{\partial t} = g(u,\upsilon) \tag{2.5}$$

If we search for the equivalence group of this system such that all the coefficients of the operator can depend on the arbitrary elements, we must add to the operators from [8] one more $p\partial_u + x\partial_t$ corresponding to the transformation

$$u' = ap(u,\upsilon) + u, \quad \upsilon' = \upsilon, \quad x' = x, \quad t' = t + ax, \quad g' = g, \quad p' = p$$

Returning to system (1.1) for its group classification we will seek the equivalence transformation operators in the form

$$X^e = \xi^t\partial_t + \xi^x\partial_x + \xi^y\partial_y + \zeta^u\partial_u + \zeta^\upsilon\partial_\upsilon + \zeta^w\partial_w + \zeta^\rho\partial_\rho + \zeta^p\partial_p + \zeta^A\partial_A \tag{2.6}$$

Unlike [7], a dependence on the arbitrary element A in all the coordinates of the infinitesimal operator X^e is admissible. Since $A = A(p, \rho)$, the operator (2.6) must satisfy the conditions of invariance of system (1.1), supplemented by the equations

$$A_t = A_x = A_y = A_u = A_\upsilon = A_w = 0 \tag{2.7}$$

The coordinates of the extended operator

$$\overline{X}^e = X^e + \zeta^{u_t}\partial_{u_t} + \zeta^{u_x}\partial_{u_x} + \zeta^{u_y}\partial_{u_y} + \ldots \tag{2.8}$$

are found from the formulae

$$\zeta^{h_\lambda} = D_\lambda^e\zeta^h - h_t D_\lambda^e\xi^t - h_x D_\lambda^e\xi^x - h_y D_\lambda^e\xi^y$$

$$D_\lambda^e = \partial_\lambda + u_\lambda\partial_u + \upsilon_\lambda\partial_\upsilon + w_\lambda\partial_w + \rho_\lambda\partial_\rho + p_\lambda\partial_p + (A_\rho\rho_\lambda + A_p p_\lambda)\partial_A$$

$$(h = u, \upsilon, w, \rho, p; \quad \lambda = t, x, y)$$

The coordinates of the extended operator (2.8), connected with the arbitrary element, by virtue of (2.7) are found from the formulae

$$\zeta^{A_\lambda} = \tilde{D}^e_\lambda \zeta^A - A_\rho \tilde{D}^e_\lambda \zeta^\rho - A_p \tilde{D}^e_\lambda \zeta^p \quad (\lambda = t, x, y)$$

$$\zeta^{A_h} = \tilde{D}^e_h \zeta^A - A_\rho \tilde{D}^e_h \zeta^\rho - A_p \tilde{D}^e_h \zeta^p \quad (h = u, \upsilon, w, \rho, p)$$

$$\tilde{D}^e_\lambda = \partial_\lambda \quad (\lambda = t, x, y), \quad \tilde{D}^e_h = \partial_h \quad (h = u, \upsilon, w)$$

$$\tilde{D}^e_\rho = \partial_\rho + A_\rho \partial_A, \quad \tilde{D}^e_p = \partial_p + A_p \partial_A$$

The group of transformations constructed using operator (2.6), which is allowed by Eqs (1.1) and (2.7), converts system (1.1) while preserving its differential structure and only changing the arbitrary element A.

For system (1.1) the equivalence group is identical with the classical one, when it is assumed that only the coefficients of the infinitesimal operator for the derivatives of the arbitrary elements [7] depend on the arbitrary elements. It is generated by the operators

$$\partial_x, \partial_y, \quad t\partial_x + \partial_u, \quad t\partial_y + \partial_\upsilon$$

$$x\partial_y - y\partial_x + u\partial_\upsilon - \upsilon\partial_u, \quad \partial_t, \quad t\partial_t + x\partial_x + y\partial_y$$

$$x\partial_x + y\partial_y + u\partial_u + \upsilon\partial_\upsilon - 2\rho\partial_\rho, \quad \rho\partial_\rho + p\partial_p + A\partial_A, \quad \partial_p, \quad \phi(w)\partial_w$$

Note. As will be seen below, in the admissible algebra for an arbitrary function $A(p, \rho)$ there is an operator $\phi(w, S)\partial_w$, where S is a certain function which depends on p and ρ and satisfies the equation

$$\rho S_\rho + A S_p = 0 \tag{2.9}$$

This operator is also obtained in an equivalence group, if we supplement it by one more arbitrary element $S = S(p, \rho)$, and we add (2.9) to Eqs (1.1) and (2.7) and

$$S_t = S_x = S_y = S_u = S_\upsilon = S_w = 0 \tag{2.10}$$

In this case, the equivalence group is only extended to the operators $\phi\partial_w$ and $F\partial_S$ with the functions $\phi = \phi(w, S)$, $F = F(w, S, p - A\ln\rho)$. The use of the algorithm in [7] to find the equivalence group for Eqs (1.1), (2.9) and (2.10) does not give these operators.

3. THE ADMISSIBLE GROUP

The operator admissible by system (1.1) can be represented in the form

$$X = \xi^t\partial_t + \xi^x\partial_x + \xi^y\partial_y + \zeta^u\partial_u + \zeta^\upsilon\partial_\upsilon + \zeta^w\partial_w + \zeta^\rho\partial_\rho + \zeta^p\partial_p$$

Integration of the defining equations reduces to solving the following equations

$$2(c_8 - c_3 + tc_4)\rho\frac{\partial A}{\partial \rho} + \sigma_2\frac{\partial A}{\partial p} + \sigma_1\left(\rho\frac{\partial A}{\partial \rho} + p\frac{\partial A}{\partial p} - A\right) = 0$$

$$\frac{\partial \sigma_2}{\partial t} = 2c_4(2p - A)$$

Here

$$\xi^t = c_4t^2 + c_8t + c_9,\quad \zeta^w = \phi(w,S),\quad \xi^x = c_1t - c_2y + c_3x + c_4tx + c_6$$

$$\xi^y = c_5t + c_2x + c_3y + c_4ty + c_7,\quad \zeta^u = -c_2\upsilon + c_3u + c_4x - 2c_4tu - c_8u$$

$$\zeta^\upsilon = c_2u + c_3\upsilon + c_4y - 2c_4t\upsilon - c_8\upsilon$$

$$\zeta^\rho = -\rho(\sigma_1 + c_8 + 2c_4t),\quad \zeta^p = p\sigma_1(t) + \sigma_2(t),\quad \sigma_1 = -4c_4t + c_{10}$$

The kernel of the fundamental Lie algebra is made up of the operators

$$X_w = \phi(w,S)\partial_w,\quad X_1 = \partial_x,\quad X_2 = \partial_y$$

$$X_4 = t\partial_x + \partial_u,\quad X_5 = t\partial_y + \partial_\upsilon$$

$$X_9 = x\partial_y - y\partial_x + u\partial_\upsilon - \upsilon\partial_u,\quad X_{10} = \partial_t,\quad X_{11} = t\partial_t + x\partial_x + y\partial_y$$

Here we have retained the numbering of the operators from [5], and S is the entropy, i.e. a certain function which depends on p and ρ and satisfies Eq. (2.9).

The group classification of system (1.1) is identical with the group classification of plane-parallel gas flows [3]. The distinctive feature of the group properties of system (1.1) is the fact that it allows the additional operators $X_w = \phi\partial_w$ with the arbitrary function $\phi(w, S)$, which make up the centre. The admissible Lie algebra is decomposed into the direct sum $L = L_7 \oplus L_\infty^w$, where L_∞^w consists of the operators X_w. Extension of the kernel of the main Lie algebras occurs by specializing the function $A(p, \rho)$.The results of the group classification, apart from the forms of the extending operators, are identical with Table 1 from [5]. In it the extending operators must be assumed to be

$$Y_1 = t\partial_t - u\partial_u - \upsilon\partial_\upsilon + 2\rho\partial_\rho,\quad Y_2 = \rho\partial_\rho + p\partial_p,\quad Y_3 = \partial_p$$

$$Y_4 = t^2\partial_t + tx\partial_x + ty\partial_y + (x - tu)\partial_u + (y - t\upsilon)\partial_\upsilon - 2t\rho\partial_\rho - 4tp\partial_p$$

$$Y_\phi = \rho\phi'(p)\partial_\rho + \phi(p)\partial_p$$

with arbitrary function $\phi(p)$.

Note that the factor algebra of the normalizer of the operator X_3 and L_{11} with respect to X_3 $(Nor_{L_{11}}(X_3)/X_3)$ consists of operators (here and below we will only indicate the number of the operators): (1, 2, 3, 4, 5, 6, 9, 10, 11), which in the invariants of the operator X_3 take the values (we have only written the changeable operators)

$$X_6 = \partial_w,\quad X_{11} = t\partial_t + x\partial_x + y\partial_y$$

4. THE OPTIMUM SYSTEM OF SUB-ALGEBRAS

The difficulty in constructing the optimum system of dissimilar sub-algebras of the algebra L is due to the presence of the operator X_w. The universal invariant of an infinite-dimensional sub-algebra of the algebra L is independent of w, and hence there are no invariant solutions of system (1.1) for it, while the partially invariant solutions are determined by its finite-dimensional sub-algebra L_k of the algebra $L_7 = \{1, 2, 4, 5, 9, 10, 11\}$. Hence, from the point of view of constructing invariant and partially invariant solutions of system (1.1) we need to investigate the finite-dimensional sub-algebras of algebra L.

If the function ϕ in the operator X_w depends only on w, the finite-dimensional sub-algebras H_m of the algebra $L = L_7 \oplus L_\infty^w$ apart from similitude, are sub-algebras of the algebra $L_7 \oplus \{\partial_w, w\partial_w, w^2\partial_w\}$. The proof of this follows from Lie's theorem [9] on the finite-dimensional sub-algebras on a straight line.

From the point of view of constructing solutions we are interested only in sub-algebras of L_7, since the solutions that are invariant and partially invariant with respect to the sub-algebras of $L_7 \oplus \{\partial_w,\ w\partial_w,\ w^2\partial_w\}$ are a special case of invariant and partially invariant solutions with respect to the sub-algebras of L_7. This follows from the fact that the operator $X_w = \phi(w)\partial_w$ from the centre and w do not occur in the other coordinates of the operator.

Another fact which enables us to ignore the infinite-dimensional part of the permitted algebra L is connected in this case with the trivial group stratification of system (1.1) into a resolving and autonomous system. For this the space of the dependent variables is extended by one more function S, namely, the entropy. Then, using the zero-order invariants [7] t, x, y, U, V, ρ, p a resolving system is obtained consisting of the equations of system (1.1) without the third equation, while the automorphous system has two equation $d_1W = 0$, $d_1S = 0$. Here the automorphous system is characterized by the fact that any of its solutions can be obtained from one non-degenerate solution [7]. Hence, to find the solutions of system (1.1), obtained from the group analysis, it is sufficient merely to construct the optimum system of sub-algebras $\theta^{(0)}$ of algebra L_7.

An algorithm for constructing a normalized optimum system is given in [5]. Here, to obtain the optimum system $\theta^{(0)}$ we used the composition series

$$0 \subset \{1,2\} \subset \{1,2,4,5\} \subset \{1,2,4,5,9\} \subset \{1,2,4,5,9,10\} \subset L_7$$

As a result of constructing the optimum system $\theta^{(0)}$ of sub-algebras of algebra L_7 we obtain that it is part of the optimum system of algebra L_{11}, constructed in [5].

It should be noted that the optimum system of sub-algebras of algebra L_7 was previously constructed in [3]. However, the optimum system obtained there does not satisfy the requirement of normalizability, and the series of sub-algebras

$$\{2, 5+\beta 1, 4+10\} \quad (\beta(\beta-1) \neq 0) \tag{4.1}$$

are missed and there are similar ones.

5. SOME SOLUTIONS OF SYSTEM (1.1)

As was established above, the basic solutions of system (1.1) are the solutions describing the plane-parallel motions of a gas. All these invariant solutions of system $(1.1)_{w=0}$ for an arbitrary function $A(p,\ \rho)$ are given in [3]. Some partially invariant solutions are also considered there. Partially invariant solutions of rank 1 and defect 1 were investigated in [4, 10]. Since the optimum system from [3] was used in [4], the solutions obtained using sub-algebras (4.1) were missed. We will fill this gap below.

The partially invariant solutions of rank 1 and defect 1 of (4.1) have the representation

$$u = t + U(\rho), \quad \upsilon = (x - t^2/2)/\beta + V(\rho), \quad p = P(\rho)$$

In order not to obtain a contradiction and to avoid a reduction to invariant solutions, it is necessary to assume

$$P' = \rho(U'^2 + V'^2) = A/\rho, \quad \beta U' + UV' = 0 \tag{5.1}$$

$$\rho_t + u\rho_x + \upsilon\rho_y + \rho(U'\rho_x + V'\rho_y) = 0, \quad \rho V'(V'\rho_x - U'\rho_y) = -1 \tag{5.2}$$

Equations (5.2) must form a complete system, otherwise all the first derivatives of the dependent functions are determined and the partially invariant solution, by the theorem in [7], is reduced to an invariant solution. Hence, we obtain

$$V = \beta \ln \frac{b\rho}{\rho + a}, \quad U = c\left(1 + \frac{a}{\rho}\right) \quad (a \neq 0)$$

and the function $\rho(t, x, y)$ is found implicitly from the relation $\Phi(\xi_1, \xi_2) = 0$ with arbitrary function $\Phi(\xi_1, \xi_2)$. Here $\xi_1 = x - t^2/2 - ct - (\rho V)'$, $\xi_2 = y + (t^3 + ct^2 - 2tx)/2\beta) - t(\rho V)' - \beta U - V$, a, b and c are arbitrary constants $a \neq 0$, and the prime denotes differentiation with respect to ρ. These solutions do not exist for an arbitrary equation of state, but only those for which $A(P(\rho), \rho) = a^2(\beta^2/(\rho + a)^2 + c^2/\rho^2)$, where $P(\rho) = \beta^2(\ln(\rho/\rho + a) + a/(\rho + a)) - 2a^2c/\rho^2$.

Note. Since $P' = A/\rho$, the flows considered are isentropic. System (5.1), (5.2) allow a transformation of equivalence with operator

$$t\partial_t + 2(x\partial_x + y\partial_y) + u\partial_u + \upsilon\partial_\upsilon - \rho\partial_\rho + p\partial_p + A\partial_A + \beta\partial_\beta$$

Using it we can obtain $\beta = 1$. This transformation is connected with the presence of an external orthomorphism L_7, corresponding to the operator

$$t\partial_t - u\partial_u - \upsilon\partial_\upsilon \tag{5.3}$$

By means of the external automorphism (5.3) the series of sub-algebras (4.1) with $\beta \neq 0$ can be reduced to {2.5 + 1.4 + 10} (verbal communication from L. V. Ovsyannikov).

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REFERENCES

1. OVSYANNIKOV L. V., *Lectures on the Fundamentals of Gas Dynamics*, Nauka, Moscow, 1981.
2. IBRAGIMOV N. Kh., Classification of invariant solutions of the equations of the two-dimensional unsteady motion of a gas. *Prikl. Mekh. Tekh. Fiz.* 4, 19–22, 1966.
3. IBRAGIMOV N. Kh., *Group Properties of Some Differential Equations.* Nauka, Novosibirsk, 1967.
4. MEN'SHIKOV V. M., Solutions of the equations of two-dimensional gas dynamics of the simple-wave type. *Prikl. Mekh. Tekh. Fiz.* 3, 129–134, 1969.
5. OVSYANNIKOV L. V., The PODMODELI program. Gas dynamics. *Prikl. Mat. Mekh.* **58**, 4, 30–55, 1994.
6. OVSYANNIKOV L. V., The PODMODELI program. *Inst. Gidrodinamiki Sib. Otd. Ross. Akad.* Nauk, Novosibirsk, 1992
7. OVSYANNIKOV L. V., *Group Analysis of Differential Equations.* Nauka, Moscow, 1978.
8. IBRAGIMOV N. H. and TORRISI M., A simple method for group analysis and its application to a model of detonation. *J. Math. Phys.* **33** 11, 3931–3937, 1992.
9. CHEBOTAREV N. G. *The Theory of Lie Groups.* Gostekhizdat, Moscow, 1940.
10. YANENKO N. N., Travelling waves of a system of quasilinear equations. *Dokl. Akad. Nauk SSSR* **109**, 1, 44–47, 1956.

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